Solution to
2021~2022 International Mathematics Assessment for Schools
Round 1 of Junior Division

1. The equation $n = 456 - 3T$ is used to define the relationship between the number of hot chocolate cups n sold per day in a coffee shop and the average daily temperature T, in degrees Celsius. According to the model, what does the number "3" mean in the equation?

   (A) For every increase of $3°C$, one more cup of hot chocolate will be sold;
   (B) For every decrease of $3°C$, one more cup of hot chocolate will be sold;
   (C) For every increase of $1°C$, 3 more cups of hot chocolate will be sold;
   (D) For every decrease of $1°C$, 3 more cups of hot chocolate will be sold;
   (E) For every decrease of $1°C$, 3 times of cups of hot chocolate will be sold.

【Suggested Solution】

From the equation :

(A) When T is increase by $3°C$, then $n = 456 - 3(T+3) = 456 - 3T - 9$, will sold 9 cups less of hot chocolate.

(B) When T is decrease by $3°C$, then $n = 456 - 3(T-3) = 456 - 3T + 9$, will sold 9 cups more of hot chocolate.

(C) When T is increase by $1°C$, then $n = 456 - 3(T+1) = 456 - 3T - 3$, will sold 3 cups less of hot chocolate.

(D) When T is decrease by $1°C$, then $n = 456 - 3(T-1) = 456 - 3T + 3$, will sold 3 cups more of hot chocolate.

(E) When T is decrease to $-304°C$, then $n = 456 - 3(-304) = 456 + 912 = 456 \times 3$, but it is unreasonable.

Answer：(D)

2. What is the sum of all positive integers n that satisfy the inequality $\frac{7}{18} < \frac{n}{5} < \frac{20}{7}$?

   (A) 25　(B) 56　(C) 78　(D) 104　(E) 105

【Suggested Solution 1】

From the given information, we get

$$\begin{cases} \dfrac{n}{5} > \dfrac{7}{18} & (1) \\ \dfrac{n}{5} < \dfrac{20}{7} & (2) \end{cases}$$

From $(1) \times 5 \times 18$, we get $18n > 35$. That is, $n > 1\frac{17}{18}$.

From $(2) \times 5 \times 7$, we get $7n < 100$. That is $n < 14\frac{2}{7}$.

It follows $1\frac{17}{18} < n < 14\frac{2}{7}$, this implies $2 \le n \le 14$.

Hence, the sum of all the positive integers n is $\dfrac{(2+14)\times 13}{2} = 104$.

【Suggested Solution 2】

Mutiply $18\times5\times7$ to inequality $\frac{7}{18}<\frac{n}{5}<\frac{20}{7}$. Then we have $245<126n<1800$.

It follows $1\frac{119}{126}<n<14\frac{36}{126}$, this implies $2\le n\le 14$.

Hence, the sum of all the positive integers n is $\frac{(2+14)\times13}{2}=104$.

Answer：(D)

3. In the given grid, an ant begins at the "Start" cell and in each move, it can go either one step right or down to any adjacent cell with common side until it reaches the "End" cell. If in each move, it gets the number in the square, then what is the smallest sum the ant can get?

Start 	18	15	13	14
20	21	12	26	21
17	7	11	19	10
6	8	17	24	11
10	20	4	19	**End**

(A) 91 (B) 95 (C) 96

(D) 97 (E) 102

【Suggested Solution】

We filled the smallest sum that can reach the square on the grid, as shown in the grid below left.

Start	18	33	46	60
20	39	45	71	81
37	44	55	74	84
43	51	68	92	95
53	71	72	91	**End**

Start	18	15	13	14
20	21	12	26	21
17	7	11	19	10
6	**8**	**17**	24	11
10	20	**4**	**19**	**End**

Then we find the smallest sum to reach the finish is 91, the path as grid shown above right.

Answer：(A)

4. A set of 13 positive integers have a mean of 8 and a median of 9. What is the greatest possible integer that is found in this set? (Note: The *mean* of the set is the sum of the all numbers of the set divided by the count of numbers of the set. The *median* of the set is the "middle" number, when the all numbers of the set are listed in order from smallest to greatest.)

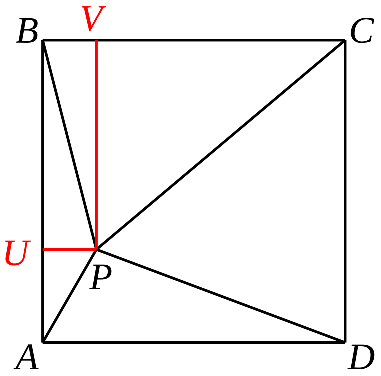
(A) 26 (B) 32 (C) 38 (D) 42 (E) 44

【Suggested Solution】

Since we want to find the greatest possible positive integer found in this set, we need to use the smallest possible integers of the first six numbers, so we will use six 1's. The 7th number (when arranging the numbers smallest to largest) must be 9 since the median is 9, because of this the 8th until the 12nd number must be 9 as well to satisfy our conditions. So the largest possible integer is just $8\times13-6-9\times6=44$.

Answer：(E)

5. In the diagram below, $ABCD$ is a square and P is a point inside it. It is known that $PA=1$cm, $PB=2$cm and $PC=3$cm, what is the length, in cm, of PD?



(A) $\sqrt{2}+\sqrt{3}$ (B) $\sqrt{6}$ (C) 4 (D) $\sqrt{17}$ (E) 6

【Suggested Solution】

Suppose P projects to U on AB and V on BC. Then by Pythagoras theorem, we have $PA^2=AU^2+BV^2$, $PB^2=BU^2+BV^2$, $PC^2=CV^2+BU^2$, $PD^2=AU^2+CV^2$. So $PD^2=PA^2+PC^2-PB^2=6$, i.e. $PD=\sqrt{6}$ cm.

Answer：(B)

6. Let $\overline{ab}$ be a two-digit number. Now, swap its digits to form another two-digit number, multiply it by 5 and take the remainder of the result when divided by 9. If the resulting number is 5, then how many different $\overline{ab}$ exist?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

【Suggested Solution】

The reminder of $5\times\overline{ba}$ when divided by 9 is 5. Therefore the reminder of $a+b$ when divided by 9 is 1. So $a+b=1$ or 10.

If $a=1$, then $b=0$ or 9. The former case is impossible as $\overline{ba}$ is also a 2-digit number. So $b=9$.

If $a=2$, then $b=8$. If $a=3$, then $b=7$. If $a=4$, then $b=6$. If $a=5$, then $b=5$. If $a=6$, then $b=4$. If $a=7$, then $b=3$. If $a=8$, then $b=2$. If $a=9$, then $b=1$. Altogether 9 possibilities.

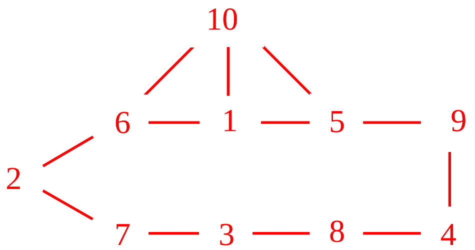
Answer：(C)

7. At most how many numbers can you choose from the set {1, 2, …, 10} such that the positive difference of any two chosen numbers is not 4, 5 or 9?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

【Suggested Solution】

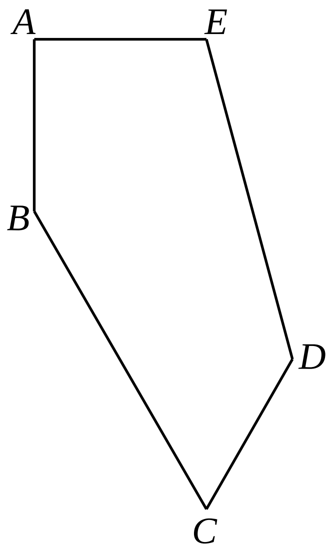
Write down the ten numbers and connect any two numbers with differences in {4, 5, 9}, one can get the following graph.

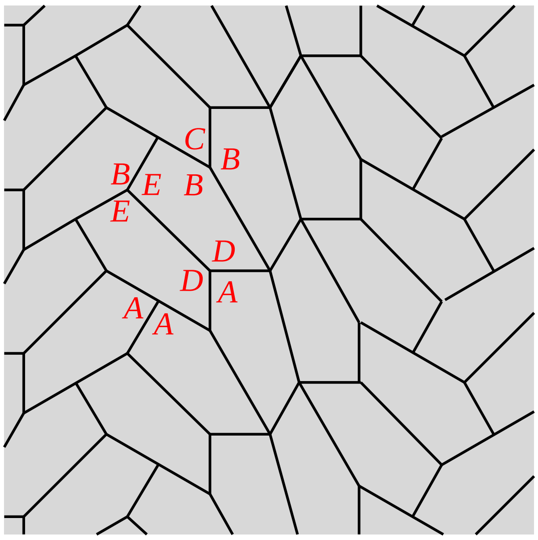


If 10 is selected, then 1, 5, 6 are not. Since adjacent numbers can not be both selected, at most three of 2, 7, 3, 8, 4, 9 are selected, giving 4 selected numbers.
If 10 is not selected, then at most 4 among the remaining 9 numbers are chosen.
In summary, there are at most 4 numbers chosen, for example, {1, 2, 3, 4}.

Answer：（A）

8. As show in the diagram below, we use some number of pentagons, that is identical to *ABCDE* to fill the plane below. What is the angle measure, in degrees, of $\angle ABC$?





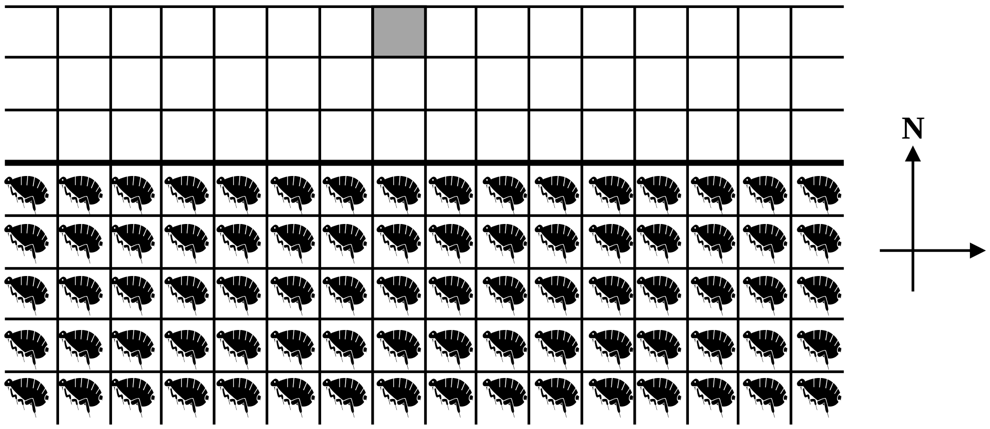
（A）100°　　（B）105°　　（C）120°　　（D）135°　　（E）150°

【Suggested Solution】

We notice that twice of $\angle A$ is $180°$, so $\angle A = 90°$. Twice of $\angle D$ and $\angle A$ is a whole $360°$, so $\angle D = \dfrac{360° - 90°}{2} = 135°$. We also find that $\angle C$ and twice of $\angle B$ is a whole $360°$, that is, $\angle C + 2\angle B = 360°$. Similarly, $\angle B + 2\angle E = 360°$. We also have the sum of all angles is $540°$, so $\angle B + \angle C + \angle E = 315°$. Solve these equations, we get $\angle B = 150°$, $\angle C = 60°$ and $\angle E = 105°$.

Answer：（E）

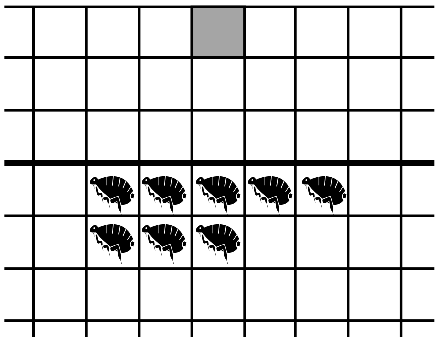
9. The world of the beetles consists of one entire plane divided into unit squares. Initially, all the beetles are located at squares south of an inner wall and each square is occupied by at most one beetle. In each move, a beetle can jump over another beetle in an adjacent square and land on the square immediately beyond. The beetle that was jumped over is then removed. However, the move is not permitted if the square is already occupied. The jump may be northward, eastward or westward. At least how many beetles are needed so that one beetle reaches the shaded square in the diagram below?



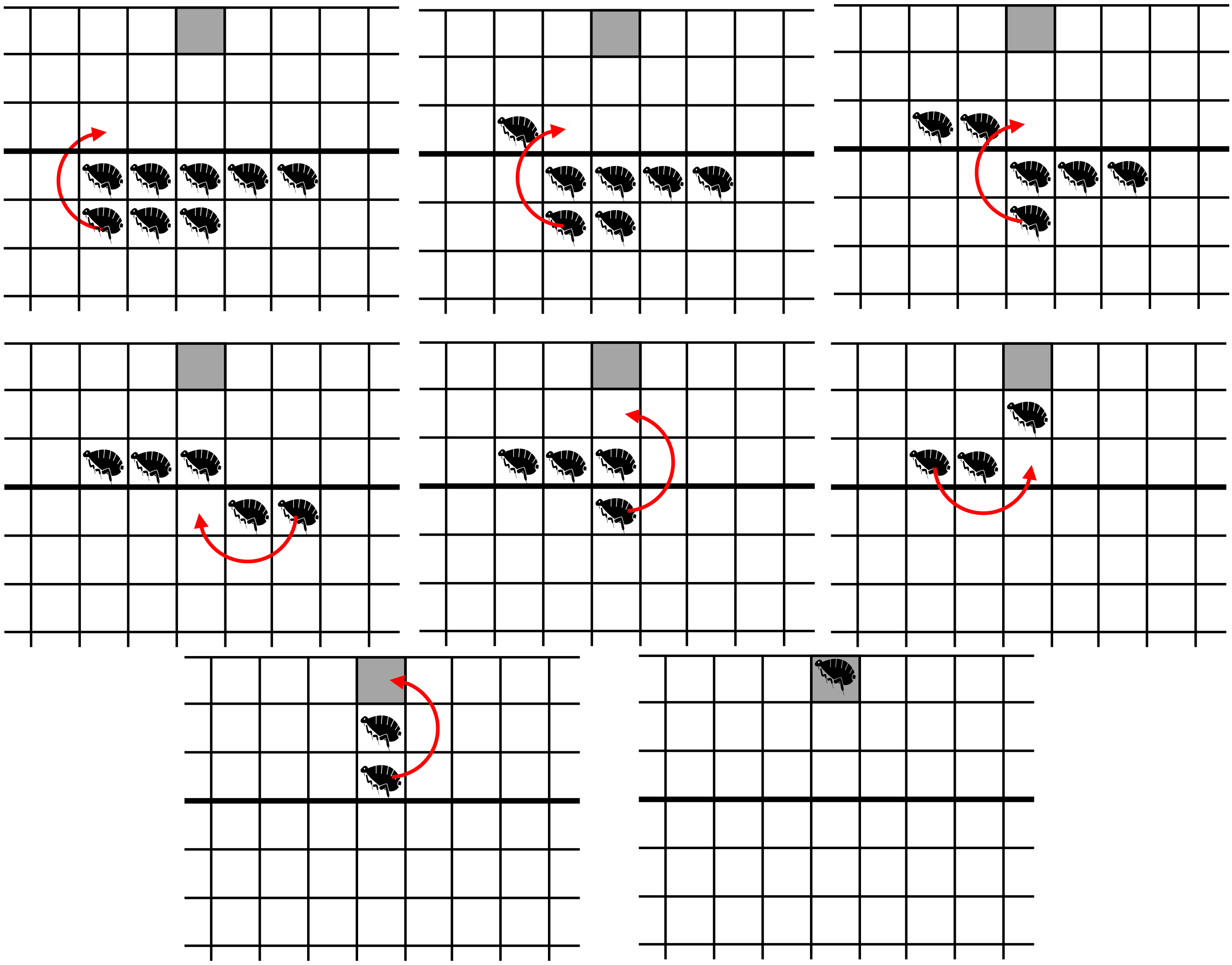
（A）5　　（B）6　　（C）7　　（D）8　　（E）10

【Suggested Solution】

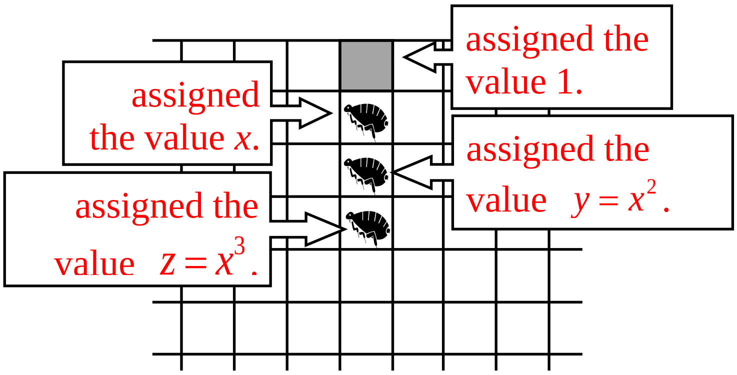
Eight beetles are enough, positioned as shown in the diagram on the right.

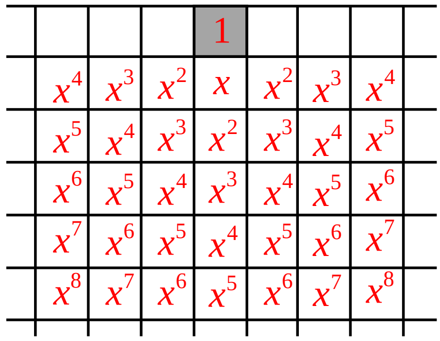


After the 7 moves, one beetle will reach the shaded square:



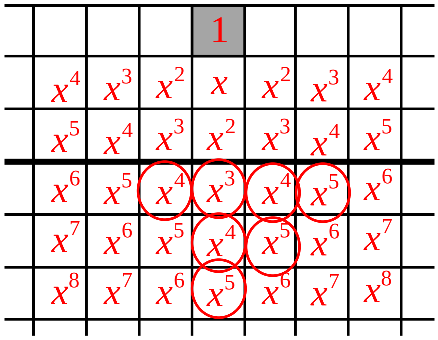
Assign x to that beetle and y to the beetle before making the jump, assigned the value of shaded square be 1. Since $x+y=1$, we may have $y=x^2$. A beetle with value z could jump over the one of value y to become the one with value x. Let $z=x^3$.





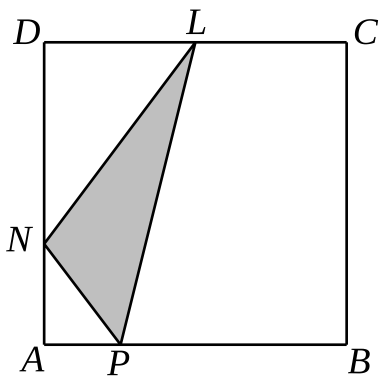
A team of size seven is insufficient, because the maximum total value of the beetles is

$$x^3+3x^4+3x^5<x^3+4x^4+2x^5=3x^3+2x^4=2x^2+x^3=x+x^2=1.$$



Answer：（D）

10. In the diagram below, $ABCD$ is a square with side length 12 cm. Point P is on side AB such that AP:PB = 1:3, point L is on side CD such that $CL=DL$ and point N is on side DA such that DN:NA = 2:1. What is the area, in cm^2, of the triangle PNL?



（A） 24 cm^2 （B） 30 cm^2 （C） 48 cm^2
（D） 90 cm^2 （E） 144 cm^2

【Suggested Solution】

Since AP:PB = 1:3, $AP=12\times\frac{1}{4}=3$ cm and $BP=12\times\frac{3}{4}=9$ cm.

Since $CL=DL$, $CL=DL=12\times\frac{1}{2}=6$ cm.

Since DN:NA = 2:1, $AN=12\times\frac{1}{3}=4$ cm and $DN=12\times\frac{2}{3}=8$ cm.

Then the area of right triangle APN, DNL and trapezoid $APLD$ is $\frac{4\times3}{2}=6$ cm^2, $\frac{6\times8}{2}=24$ cm^2 and $\frac{(3+6)\times12}{2}=54$ cm^2, respectively.

Hence, the area of triangle PNL is $54-6-24=24$ cm^2.

Answer：（A）

11. What is the sum of all possible positive integers a, such that for any positive integer $n>100$, $\dfrac{n(n+2)(n+4)}{a}$ is an integer?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 6

【Suggested Solution】

Denote $f(n)=n\times(n+2)\times(n+4)$. Since a divides $f(101)=101\times103\times105$ and $f(102)=102\times104\times106$, then a must divide the great common divisor of $f(101)$ and $f(102)$, which is $(101\times103\times105, 102\times104\times106)=(105, 102\times104)=3$.

But for any $n>100$, $f(n)=n\times(n+2)\times(n+4)\equiv n\times(n+2)\times(n+1)\equiv 0 \pmod 3$, since the product of three consecutive integers is a multiple of 3.

So, the sum of all positive integers a is $a=1+3=4$.

Answer: (D)

12. The distance between Frankfurt and New York is 6750 km and the time difference between both cities is 6 hours. If it is 08:00 in New York right now, then it is 14:00 already in Frankfurt. At 15:00 Frankfurt time, an Airbus 320 with a cruising speed of 700 km/h departs from Frankfurt to New York. While at 10:00 New York time, a Boeing 787 plane with a cruising speed of 800 km/h departs from New York to Frankfurt. The route goes over the Atlantic Ocean and it is known that there is a constant wind blowing from New York to Frankfurt at a speed of 100 km/h. It is also known that the Airbus 320 cannot fly more than 5 hours without re-fueling, while the Boeing 787 can fly for over 12 hours without re-fuelling. The Airbus 320 will need to re-fuel at a small airport, which is 2400 km from Frankfurt. The re-fuelling and take-off will require 1 hour in total. What will be the time in New York when the two planes meet assuming they fly the same trajectory?

(A) 13:30 (B) 14:30 (C) 15:30 (D) 16:30 (E) 20:30

【Suggested Solution】

As the question asks what will be the time in New York when the planes meet let's use only New York time. The Airbus 320 departs at 15:00 German time, this is 09:00 am. NYC time. The Boeing 787 departs at 10:00 am, i.e1 hour later. Since the Airbus 320 will takes 1 hour for re-fuelling, we may consider as the two planes take off at the same 10:00 am New York time and without re-fuelling.

They have a combined speed of 1500 km/h (one can ignore the wind because it increases the speed of the Boeing 787 and reduces the speed of the Airbus 320). So both planes will meet $6750\div1500=4.5$ hours after 10:00 am. New York time, that is at 14:30 New York time.

Answer: (B)

13. What is the remainder when $1^2+2^2+3^2+....+2020^2+2021^2$ is divided by 4?

(A) 2021 (B) 1011 (C) 1 (D) 2 (E) 3

【Suggested Solution】

Since the square of any positive odd integer when divided by 4 the remainder is 1, and the square of any positive even integer when divided by 4 the remainder is 0.

Since there are $(2021+1)\div 2=1011$ odd integers from 1 to 2021, thus the total sum of the remainder of 1^2, 2^2, 3^2, …, 2020^2, 2021^2 divided by 4 is 1011.
We have $1011=4\times 252+3$, hence the remainder of $1^2+2^2+3^2+....+2020^2+2021^2$ when divided by 4 is 3.

Answer：(E)

14. If $a=\sqrt[3]{4}+\sqrt[3]{2}+\sqrt[3]{1}$, then what is the value of $\frac{3}{a}+\frac{3}{a^2}+\frac{1}{a^3}$?

(A) 1 (B) 7 (C) $\sqrt[3]{6}$ (D) $\frac{19}{8}$ (E) $\sqrt[3]{4}$

【Suggested Solution】

From the identity $x^3-y^3=(x-y)(x^2+xy+y^2)$, let us assume $\sqrt[3]{2}=m$, then $m^3=2$, $a=m^2+m+1$.

It follows $(m-1)a=(m-1)(m^2+m+1)=m^3-1=1$, then $\frac{1}{a}=m-1$. Therefore,

$$\frac{3}{a}+\frac{3}{a^2}+\frac{1}{a^3}=3(m-1)+3(m-1)^2+(m-1)^3$$
$$=(m-1)(3+3(m-1)+(m-1)^2)=(m-1)(m^2+m+1)=m^3-1=1$$

Answer：(A)

15. Let *a* be a positive integer, where the positive difference of every two digits of *a* in base 10 are all written on a board. If some of the numbers are erased and only the numbers 2, 0, 2 and 2 are left, what is the least possible value of *a*?

(A) 1011 (B) 1112 (C) 1113 (D) 2000 (E) 2022

【Suggested Solution】

We know that *a* is at least 4 digits. At least two of them are equal, say *x*, *x*, which give a difference of 0. Another digit has difference 2 with *x*, so it is $x+2$ or $x-2$. The last digit should have difference 2 with one of previous three. In summary, we have $(x, x, x-2, x-4)$; $(x, x, x-2, x)$; $(x, x, x-2, x+2)$; $(x, x, x-2, x-2)$; $(x, x, x, x+2)$; $(x, x, x+2, x+4)$ or equivalent solutions.
So the least possible value of *a* is 1113.

Answer：(C)

16. There are three integers *x*, *y* and *z* such that $x\le y\le z\le 8$, $x+y+z=12$ and $xy+yz+zx=27$. What is the value of *xyz*?

(A) 40 (B) 8 (C) 0 (D) –8 (E) –40

【Suggested Solution】

It is obvious that $4\le z$. Use *z* to represent *x*, *y* and get

$$x+y=12-z$$
$$xy=27-(x+y)z=27-(12-z)z=z^2-12z+27$$

Given $x+y$ and *xy* as integers, there are integer solutions *x*, *y* if and only if the discriminant is a perfect square, which is

$$(12-z)^2-4(z^2-12z+27)=36+24z-3z^2=3(12+z(8-z)).$$

Plug in $z \in \{4, 5, 6, 7, 8\}$ and find that $z = 5, 8$ give a perfect square. Then we get the solutions $(x, y, z) = (-1, 5, 8)$ or $(x, y, z) = (-1, 8, 5)$, and only $(-1, 5, 8)$ satisfies $x \le y \le z$, which gives $xyz = (-1)\times 5\times 8 = -40$.

Answer：（E）

17. Charlie and Danny bought some apple and carrot juices in the supermarket. Charlie bought some brand *A* apple juices which costs \$5 per bottle and some brand *B* carrot juices which costs \$1.5 per bottle, spending a total of \$139.5. Meanwhile, Danny bought some brand *C* apple juice which costs \$4.5 per bottle and some brand *D* carrot juices which costs \$2 per bottle, spending a total of \$167. If it is known the number of fruit juices that each of them bought is the same, then how many brand *A* apple juices did Charlie buy?

（A）15　　（B）18　　（C）21　　（D）25　　（E）33

【Suggested Solution 1】

When Charlie only buys carrot juices, he buys a total of 139.5 ÷ 1.5 = 93 bottles of carrot juices. If Charlie buys 3 bottles of apple juices and 10 packs of carrot juices, and each kind of fruit juice at 15 dollars, so the combination of apple juices and carrot juices that Charlie can buy are as follow

Number of bottles of Apple juices	0	3	6	9	12	15	18	21	24	27
Number of bottles of Carrot juices	93	83	73	63	53	43	33	23	13	3
Total number of bottles	93	86	79	72	65	58	51	44	37	30

When Danny tries to buy more bottles of carrot juices and assuming that he buys only 2 bottles of apple juices, then, Danny must buy $(167 - 2\times 4.5)\div 2 = 79$ bottles of carrot juices. Also, if Danny buys 9 bottles of carrot juice and 4 bottles of apple juice, both are 18 dollars, so the combination of bottles of apple juice and bottles of carrot juice that Danny can buy is as follows:

Number of bottles of Apple juices	2	6	10	14	18	22	26	30	34
Number of bottles of Carrot juices	79	70	61	52	43	34	25	16	7
Total number of bottles	81	76	71	66	61	56	51	46	41

Observe the two charts, the total number of bottles of apple juices and carrot juices are the same, which is 51. It follows Charlie bought 18 bottles of apple juices.

【Suggested Solution 2】

Let the number of fruit juices that each of them bought is *s*, Charlie bought *a* bottles brand *A* apple juices, Danny bought *c* bottles brand *C* apple juice. Then we can get: $5a + 1.5(s - a) = 139.5$ and $4.5c + 2(s - c) = 167$. Simplify it we have $7a + 3s = 279$ and $5c + 4s = 334$, cancellation *s* we get $28a - 15c = 114$. Since *a* and *c*are positive integers, then *c* must be even number, *a* must be multiple of 3 with unit digit 8.

When $a = 8$, then $15c = 110$, *c* is not a integer, it is contradiction.

When $a = 18$, then $c = 26$. Hence Charlie bought $(139.5 - 5\times 18)\div 1.5 = 33$ bottles brand *A* apple juices, Danny bought $(167 - 4.5\times 26)\div 2 = 25$ bottles brand *C* apple juice, the total number of apple juices and carrot juices are the same, which is 51.

When $a \ge 48$, then $5a \ge 240$, it is contradiction.

It follows Charlie bought 18 bottles of apple juices.

Answer：（B）

18. It is known that Annie can wash 3 plates every minute, while Betty can wash 2 plates every minute. Also, it is known that Annie can wash 9 cups every minute, while Betty can wash 7 cups every minute. If there are a total of 134 dirty plates and cups and both of them worked together in washing all the dirty plates and cups and finished in exactly 20 minutes, how many dirty plates are there?
（A）42 （B）49 （C）50 （D）60 （E）84

【Suggested Solution】

If we only rely on the conditions of the problem, it seems that this problem cannot be answered, but in fact it can be answered cleverly. We no need to consider the speed of washing plates and cups.

Assume Annie spent x minutes washing plates, then she washed a total of $3x$ dishes, and in the remaining $20-x$ minutes he washed a total of $9\times(20-x)$ cups. Suppose Betty spent y minutes washing plates, Then Betty washed a total of $2y$ plates, and used the remaining time to wash $7\times(20-y)$ cups.

Since there are 134 plates and coffee cups, we can set an equation as $3x+9(20-x)+2y+7(20-y)=134$. After simplified, we have $6x+5y=186$.

Because x and y are not more than 20, and we know that $5y$ must be a multiple of 5, so the ones digit of $5y$ is either 0 or 5, since it can't be end in 5, due to $6x+5y=186$ and 186 then is an even number, so both x and y must be even numbers. Hence, the ones digit of $5y$ must be 0. Next, consider $6x$, so the value of x can be 1, 6 or 16. When $x=1$ or 6, the value of y will exceed 20, which is a contradict.

It follows so $x=16$, and $y=18$ at the same time.

Thus, there are $9\times(20-16)+7\times(20-18)=50$ cups and $3\times16+2\times18=84$ plates.

Answer：（E）

19. There are ten tokens, 3 black and 7 white, are to be randomly placed into each of the unit squares of a 2×5 board, where each unit square can only fit exactly one token. How many ways are there to place the tokens such that no two black tokens are adjacent to each other?
（A）30 （B）36 （C）38 （D）40 （E）60

【Suggested Solution】

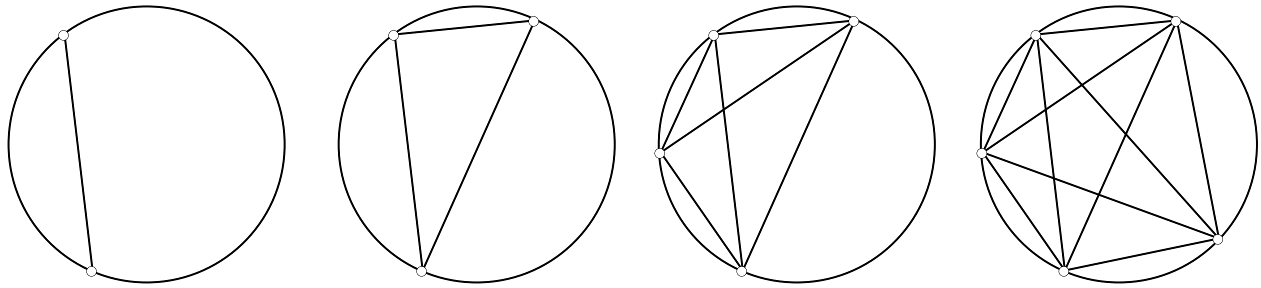
If three black tokens are in one row, they must be at positions 1, 3, 5. There are two ways for this type.

If two black tokens are in one row, the third piece in the other row. There are two ways to choose the row of two black tokens; given the row, there are $C_2^5-4=6$ ways to put these two black tokens; there are 3 ways to put the third black token. There are $2\times6\times3=36$ ways for this type.

Hence, the total ways to put the three black tokens is $2+36=38$ ways.

Answer：（C）

20. The four circles below have some number of points on their circumference. Connect all the points on the same circle using straight lines and count the number of regions these segments have partitioned the circle into.



2 points, 2 regions.   3 points, 4 regions.   4 points, 8 regions.   5 points, 16 regions.

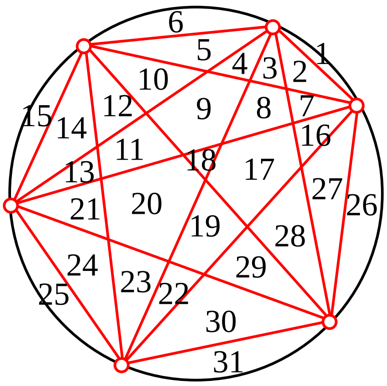
If there are 6 points on the circumference of a circle, how many regions have been partitioned at the most?

（A）24   （B）30   （C）31   （D）32   （E）40

【Suggested Solution】

Let any three segments not meet at a same point then we can get at most 31 regions as the diagram shown.

Answer：（C）



21. Given $x \neq n+0.5$, where n is an integer. Let $[x]$ represent the integer closest to x. For example, $[2.4]=2$ and $[2.6]=3$. What is the numerical value of $[\sqrt{1\times 2}]+[\sqrt{2\times 3}]+[\sqrt{3\times 4}]+\ldots+[\sqrt{40\times 41}]$?

【Suggested Solution】

Assume n is a positive integer.

From the given information, we have $n^2 < n(n+1) = (n+0.5)^2 - 0.25 < (n+0.5)^2$,

Then $n < \sqrt{n(n+1)} < n+0.5$, so that $[\sqrt{n(n+1)}]=n$.

Thus, the original expression is equal to $1+2+3+\ldots+40=820$.

Answer：820

22. A party was attended by some number of families, where each family consists of a couple and their kid(s) and it is known that a family can have at most 10 kids. It is known that one dad, one mom and one kid, all from different families, will be selected to play in a game, then there are 4884 different ways to choose. How many kids attended the party?

【Suggested Solution】

If there are n couples and totally m kids, then there are m ways to choose the kid, $n-1$ ways to choose the mum and $n-2$ ways to choose the dad.
$4884=11\times12\times37$, so $n-1=12$ or $n-1=4$. For the latter, $n=5$ and $m=407$, contradiction to that a family has at most 10 kids. So $n=13$ and $m=37$.

Answer：037

23. Two arithmetic progressions, where each corresponding term of both sequences are multiplied to each other and we obtain the following sequence: 1, 10, 100, …. Find the 6th term of the resulting sequence.

【Suggested Solution】

Write the one arithmetic progression as $a-d$, a, $a+d$, $a+2d$, … and the second arithmetic progression as $b-e$, b, $b+e$, $b+2e$, …. Then we have

$$(a-d)(b-e)=1,\ \ ab=10 \ \text{ and } \ (a+d)(b+e)=100.$$

We need to find $(a+4d)(b+4e)$. First we find that

$$ab+de=\frac{(a-d)(b-e)+(a+d)(b+e)}{2}=\frac{101}{2}$$

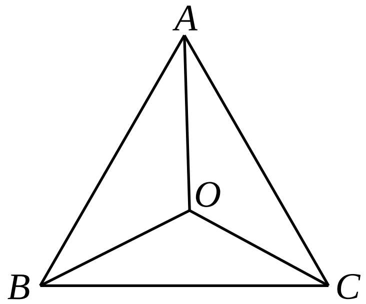
$$bd+ae=\frac{(a+d)(b+e)-(a-d)(b-e)}{2}=\frac{99}{2}$$

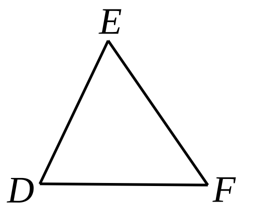
$$de=ab+de-ab=\frac{101}{2}-10=\frac{81}{2}$$

So $(a+4d)(b+4e)=ab+4(bd+ae)+16de=10+4\times\frac{99}{2}+16\times\frac{81}{2}=856$.

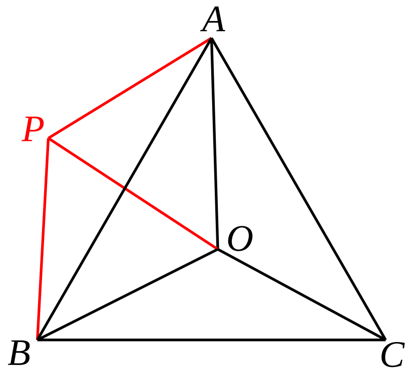
Answer：856

24. Let O be a point inside equilateral triangle ABC such that $\angle AOB=115^\circ$ and $\angle BOC=125^\circ$. If DEF is a triangle such that $EF=OA$, $FD=OB$ and $DE=OC$, what is the maximum angle measure, in degrees, of triangle DEF?





【Suggested Solution】

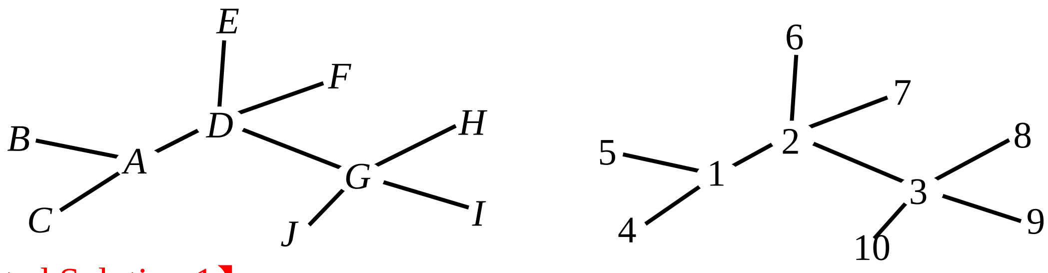


Rotate triangle OBC 60° about B to PBA. Then $PA=OC=DE$. Moreover, POB is an equilateral triangle, so that $PO=OB=FD$. Hence triangle PAO is congruent to triangle DEF.

It follows that $\angle EFD=\angle AOP=\angle AOB-60^\circ=55^\circ$, $\angle FDE=\angle OPA=\angle BPA-60^\circ=\angle BOC-60^\circ=65^\circ$ and $\angle DEF=180^\circ-\angle EFD-\angle FDE=60^\circ$.

Answer：065

25. The diagram below on the left shows a map, where cities and roads are interconnected, such that there is unique path, between any two cities, without passing any road more than once. Let x be the number of ways to labell the cities by the numbers 1, 2, …, 10 such that every path starting from city A are all strictly increasing. What is the value of $\frac{x}{20}$? The diagram below on the right shows one example.



【Suggested Solution 1】

The paths from A to H, I, J are all increasing if and only if the path from A to G is increasing and labell at G is smaller than labells at H, I, J. Among all labells of A, B, …, J, four of them satisfies that labelling of G is the smallest. Similarly, labelling of D should be the smallest among labellings of D, E, …, J; labell of A is the smallest of all labellings. So there are $\frac{10!}{10\times7\times4}=12960$ ways to labell all the cities. Hence, $\frac{a}{20}=\frac{12960}{20}=648$.

【Suggested Solution 2】

The paths from A to H, I, J are all increasing if and only if the path from A to G is increasing, labell of A is the smallest of all labellings, and only can be labelled 1. B has 9 labelling methods, and C has 8 labelling methods. The labell of D is the smallest of the remaining 7 labells, so E has 6 labelling methods, and F has 5 labelling methods. The labell of G is the smallest of the remaining 4 labells, so H has 3 labelling methods, I has 2 labelling methods, and J has 1 labelling method. So there are $9\times8\times6\times5\times3\times2\times1=12960$ ways to labell all the cities. Hence, $\frac{a}{20}=\frac{12960}{20}=648$.

Answer：648